Dispelling Commonplaces on Wave Mechanics

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ABSTRACT- The extension of exact ray-trajectories to any kind of wave-like features allows a new interpretation of Wave Mechanics, challenging 90 years of "quantum paradoxes".

1- Classical Wave Mechanics

In a paper published in 2009, and in other papers published in the following years [**1-4**], a **remarkable discovery** (concerning the possibility of **describing in terms of exact ray-trajectories any kind of waves)** was presented, and welcomed by an (even more remarkable) icy silence on the part of the Scientific Community, with few exceptions [**5-8**].

Having mainly in mind such typically wave-like features as diffraction and interference (which require, as is well known, a strict mono-chromaticity) the Authors referred, in order to fix ideas (but in full generality), to electromagnetic waves obeying a scalar wave equation of the simplest possible form

$$\nabla^2\psi - \frac{n^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \ , \qquad (1)$$

where $\psi(\vec{r}, t)$ represents any component of the electric and/or magnetic field and $n(\vec{r})$ is the (time independent) refractive index of the medium. They ensured both wave mono-chromaticity and coherence by the obvious assumption

$$\psi = u(\vec{r},\omega)e^{-i\,\omega t} , \qquad (2)$$

thus obtaining the well-known Helmholtz equation

$$\nabla^2 u + (n\,k_0)^2 u = 0 \ , \qquad (3)$$

where $k_0 \equiv \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$: a *time-independent* equation whose solution they looked for, as it's often done, in the quite general form

$$u(\vec{r},\omega) = R(\vec{r},\omega)\ e^{\,i\,\varphi(\vec{r},\omega)} , \qquad (4)$$

with real amplitude $R(\vec{r},\omega)$ and phase $\varphi(\vec{r},\omega)$. By defining then the wave-vector

$$\vec{k} = \vec{\nabla}\varphi(\vec{r},\omega) \qquad (5)$$

and the functions

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$$W(\vec{r},\omega) = -\frac{c}{2k_0}\frac{\nabla^2 R(\vec{r},\omega)}{R(\vec{r},\omega)} \tag{6}$$

and

$$D(\vec{r},\vec{k},\omega) \equiv \frac{c}{2k_0}[k^2 - (n k_0)^2] + W(\vec{r},\omega) \quad , \tag{7}$$

the Helmholtz equation turns out to be associated with a Hamiltonian equation system of the form

$$\begin{cases} \frac{d\vec{r}}{dt} = \frac{\partial D}{\partial \vec{k}} \equiv \frac{c\vec{k}}{k_0} & (8) \\ \frac{d\vec{k}}{dt} = -\frac{\partial D}{\partial \vec{r}} \equiv \vec{\nabla}[\frac{ck_0}{2} n^2(\vec{r},\omega) - W(\vec{r},\omega)] & (9) \\ \vec{\nabla}\cdot(R^2\ \vec{k}) = 0 & (10) \end{cases}$$

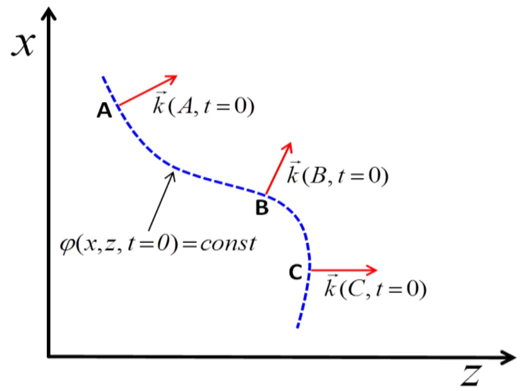
providing the full *kinematics* (both "rails" and "time-table") of a *stationary set* of ***exact ray-trajectories*** (orthogonal to the phase-surfaces), mutually *coupled* by the "*Wave Potential function*" $W(\vec{r},\omega)$ of eq.(6) and reducing to the *geometrical optics approximation* when $W(\vec{r},\omega)$ is negligible.

Contrary to the commonplace that a treatment in terms of *ray-trajectories* is only possible for a limited number of physical cases (such as reflection and refraction) ascribed to the *geometrical optics approximations*, eqs.(8)-(10) determine therefore the *stationary* frame on which an *exact*, *ray-based* description of wave-like features is always possible.

Notice, moreover, that ***no statistical concept*** was invoked**.**

The "Wave Potential" $W(\vec{r},\omega)$, inducing a mutual *perpendicular* coupling between *mono-chromatic* trajectories, *is the one and only cause of wave-like features such as diffraction and interference*. It represent a *stationary, structural property* of the Helmholtz equation, "triggered" by the wave amplitude distribution over the launching phase-plane, and determining both the *geometry* of the trajectories traveled by the rays and the *motion laws* of the ray themselves. An important consequence of its *perpendicularity* to the ray paths is the property of leaving the intensity of the ray velocity $\frac{d\vec{r}}{dt} \equiv \frac{c\vec{k}}{k_0}$ unchanged, *confining the coupling action to a mere deflection*.

The numerical time-integration of the Hamiltonian system provides, step by step, the values of $\vec{r}(t), \vec{k}(t), R(\vec{r},t)$ all over the space region spanned by the considered ray-trajectories, by assigning, on an arbitrary "launching" surface (assumed as initial phase surface $\varphi(\vec{r}, t=0) = const$) the wave amplitude distribution function $R(\vec{r}, t=0)$ - for instance, in the *2*-dimensional (*x*, *z*) case, a



Gaussian distribution of the form $R(x;z=0) \div exp(-x^2/w_0^2)$ - together with the values, at each point $\vec{r}(t=0)$, of the wave-vector $\vec{k}(t=0)$, normal to the launching surface.

Beyond its novelty within Classical Mechanics, this discovery opens the way to a basic further discovery concerning the very **interpretation of Wave Mechanics**, and challenging 90 years of "quantum paradoxes".

2- De Broglie's Wave Mechanics

Let us recall, in fact, that **Heisenberg** [**9**] had already conceived his Uncertainty Principle (rejecting the causal and realistic nature of physical laws) when **de Broglie** [**10**, **11**] proposed his seminal Ansatz

$$\vec{p}/\hbar \rightarrow \vec{k} \qquad (11)$$

associating with material particles of assigned momentum, for the first time in the history of Physics, a suitable "**matter wave**" whose **physical objective existence** was very soon confirmed by the experiments performed by **Davisson** and **Germer** [**12**] on electron diffraction by a crystalline nickel target.

We also recall that the **classical** motion of a single particle of mass m and assigned energy E, launched into a stationary potential $V(\vec{r})$, may be described by the time-independent Hamilton-Jacobi (H-J) equation

$$(\vec{\nabla} S)^2 = 2\,m\,[E - V(\vec{r})] \ , \qquad (12)$$

where the basic property of the H-J function $S(\vec{r},E)$ is that the particle momentum is given by

$$\vec{p} = \vec{\nabla} S(\vec{r},E). \qquad (13)$$

The (classical) H-J surfaces $S(\vec{r},E) = const$, in other terms, have the basic property of **"piloting" particles** along a set of **fixed trajectories**, determining also their motion laws.

De Broglie's Ansatz (11) may now be re-written in the form

$$\vec{p}/\hbar \equiv \vec{\nabla} S(\vec{r},E)/\hbar \rightarrow \vec{k} \equiv \vec{\nabla}\varphi \, , \tag{14}$$

showing that the H-J surfaces $S(\vec{r},E) = const$ are *the phase-fronts of de Broglie's matter waves*, while maintaining - *just as in Classical Mechanic* - their basic role of "**piloting**" **the particles,** according to eq.(13).

Then **Schrödinger** obtained his equations [**13**,**14**], having in mind an **exact, causal** and **realistic** description of physical nature; but **Born** [**15**] proposed at once a statistical interpretation which became the standard one, universally accepted by the scientific community. Thanks to Heisenberg's Principle the concept itself of exact trajectory-based motion laws was considered to be meaningless, and the choice of describing particles by means of packets of matter waves, whose extent (in $\vec{r}$ and $\vec{k}$ space) satisfied relations of the form

$$\begin{cases} \Delta x \, \Delta p_x \geq \hbar \\ \Delta y \, \Delta p_y \geq \hbar \\ \Delta z \, \Delta p_z \geq \hbar \end{cases} , \tag{15}$$

appeared to confirm the generality of the Uncertainty Principle.

The situation, however, **is now radically changed**: observing in fact that the time-independent **Schrödinger** equation

$$\nabla^2 u(\vec{r},E) + \frac{2m}{\hbar^2}[E - V(\vec{r})] \, u(\vec{r},E) = 0 \tag{16}$$

is itself a **Helmholtz-like** equation, we may submit it to **the same treatment applied to the classical eq.(3)**, by assuming, in analogy with eq.(4),

$$u(\vec{r},E) = R(\vec{r},E) \; e^{\, i \, S(\vec{r},E)} \tag{17}$$

with real amplitude $R(\vec{r},E)$ and phase $S(\vec{r},E)$, and defining, in analogy with eq.(6), the "**Wave Potential**" function

$$Q(\vec{r},E) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R(\vec{r},E)}{R(\vec{r},E)} \tag{18}$$

and the Hamiltonian function

$$H(\vec{r},\vec{p},E) = \frac{p^2}{2m} + V(\vec{r}) + Q(\vec{r},E) \; . \tag{19}$$

The same previous procedure leads now to the dynamical Hamiltonian system

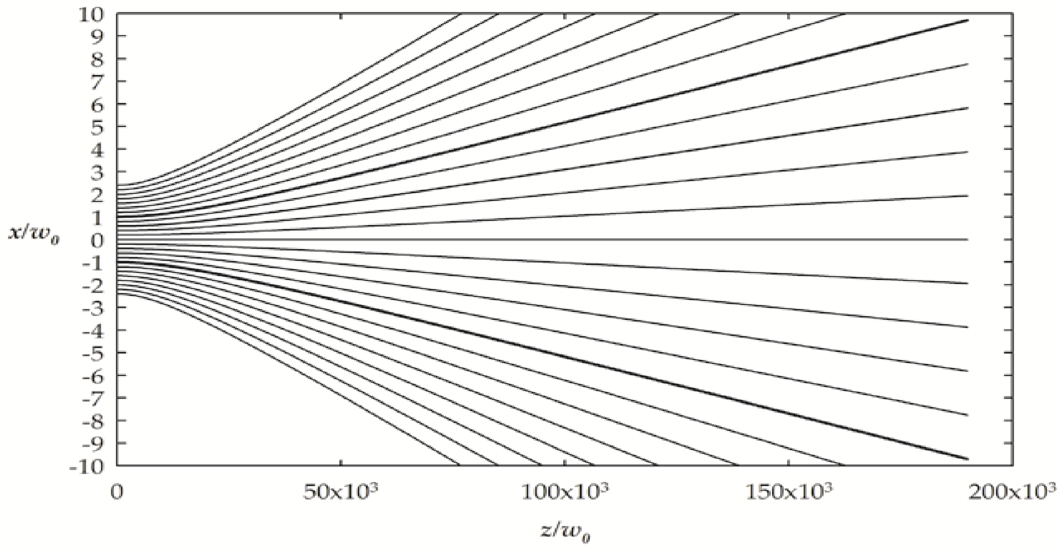
$$\begin{cases} \dfrac{d\,\vec{r}}{d\,t} = \dfrac{\partial H}{\partial \vec{p}} \equiv \dfrac{\vec{p}}{m} & (20) \\ \dfrac{d\,\vec{p}}{d\,t} = -\dfrac{\partial H}{\partial \vec{r}} \equiv -\vec{\nabla}[V(\vec{r}) + Q(\vec{r},E)] & (21) \\ \vec{\nabla}\cdot(R^2\,\vec{p}) \equiv 0 & (22) \end{cases}$$

- fully analogous to the system (8)-(10),
- providing, *without resorting to statistical concepts*, a set of **exact particle trajectories** perpendicularly coupled by the WavePotential $Q(\vec{r},E)$, and
- reducing to the classical dynamical equations when $Q(\vec{r},E)$ is absent.

Classical Dynamics constitutes therefore the *geometrical optical approximation* of **Wave Mechanical Dynamics**. Just like in Classical Mechanics, the Hamiltonian dynamical system (20)-(22) describes the motion, along a fixed set of "rails", of **point-particles**, whose representation in terms of **statistical wave-packets** is **neither required nor conceivable**.

We may notice that the wave-like behavior of matter (including interference and diffraction) doesn't directly concern particles, but their trajectories: the number of particles traveling along these "rails" is quite uninflent. The term $-\vec{\nabla}Q(\vec{r},E)$ (which we could call "**Helmholtz-de Broglie pseudo-force**") is a typically wave-like consequence of the geometrical set up.

Many examples of numerical solution of the Hamiltonian particle dynamical system (20)-(22) in cases of diffraction and/or interference were given in Refs.[**1-4**], by assuming, for simplicity sake, the absence of external fields and a geometry allowing to limit the computation to the (*x*,*z*)-plane, for waves launched along the *z*-axis.



De Broglie's wave trajectories and waist lines on the symmetry *(x,z)*-plane for a *Gaussian* beam with waist w_0 and $\lambda_0 / w_0 = 2\times 10^{-4}$.

The particle trajectories were computed, with initial momentum components $p_x(t=0)=0$; $p_z(t=0)=p_0=2\pi\hbar/\lambda_0$, by means of a *symplectic* integration method. We limit here ourselves to present in the figure the particle trajectories on the *(x,z)-plane* relevant to the diffraction of a *Gaussian* particle beam traveling along the *z-axis* and starting from a vertical slit centered at $x=z=0$ in the form $R(x;z=0)\div exp(-x^2/w_0^2)$, where the length w_0 is the so-called *waist radius* of the beam. The two heavy lines are its *waist-lines*, given by the analytical relation

$$x(z)=\pm\sqrt{w_0^{\,2}+\left(\frac{\lambda_0\, z}{\pi\, w_0}\right)^2} \qquad (23)$$

representing, in the well-known *paraxial approximation* [**16**], the trajectories starting (at $z=0$) from the *waist positions* $x=\pm w_0$. The agreement between the *analytical* expression (23) and our *numerical* results provides, of course, an excellent test of our approach and interpretation.

IN CONCLUSION, **contrary to the commonplace** that the concept itself of "particle trajectory" is devoid of physical meaning, **particle trajectories are a direct and exact consequence of the time-independent Schrödinger equation,** dispelling any statistical interpretation.
Let us point out that, as discussed in Ref.[**4**], **Bohm**'s approach [**17-23**] to quantum trajectories, viewing as a "pilot wave" the Born wave-function, combining the full set of Schrödinger's eigen-solution (**a "wave" for which no Davisson-Germer experiment was ever contrived!**), remains within the probabilistic limits of the Copenhagen interpretation.
Bohm's so called "Quantum Potential", indeed, is merely a statistical average [**4**] taken over the full set of mono-chromatic Helmholtz-de Broglie Wave Potentials.
We synthetize our approach, together with Bohm's one, in the following Tables I and II, comparing the respective sets of equations for the motion of single particles in an external potential $V(\vec{r})$:

TAB.I EXACT (POINT-PARTICLE) DESCRIPTION	TAB.II PROBABILISTIC (WAVE-PACKET) DESCRIPTION
$\frac{d\vec{r}}{dt}=\frac{\vec{p}}{m}$ $\frac{d\vec{p}}{dt}=-\vec{\nabla}[V(\vec{r})-\frac{\hbar^2}{2m}\frac{\nabla^2 R(\vec{r},E)}{R(\vec{r},E)}]$ $\vec{\nabla}\cdot(R^2\vec{p})=0$	$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V(\vec{r})\psi$ $\frac{d\vec{r}}{dt}=\frac{\hbar}{m}\,\mathrm{Im}\frac{\vec{\nabla}\psi(\vec{r},t)}{\psi(\vec{r},t)}$

The equations of TAB.I are structurally *encoded in Schrödinger's (Helmholtz-like) time-independent equation*, and provide the ***exact trajectories*** *of a point-particle* of assigned energy *E*, piloted by a *mono-chromatic de Broglie matter wave* whose objective physical reality is undeniably certified by its observed diffractive properties [**12**].
The equations of TAB.II, on the other hand, provide the ***probability flux-lines*** of a *wave-packet* (resulting from an average over the entire set of stationary eigen-solutions) built up as a whole by *the simultaneous solution of Schrödinger's time-dependent equation*, declaredly giving an equivalent version of the (intrinsically probabilistic) Copenhagen vision, including most "quantum paradoxes".

3- Conclusions The comparison between the stationary Helmholtz coupling of mono-energetic trajectories (along which point-particles are driven by de Broglie's mono-chromatic waves) and the inextricably non-local coupling among any part of the physical system involved by the "guidance equation" $\frac{d\vec{r}}{dt} = \frac{\hbar}{m} \operatorname{Im} \frac{\vec{\nabla}\psi(\vec{r},t)}{\psi(\vec{r},t)}$ leads us to conclude that the Bohmian theory is far from the spirit of our non-probabilistic approach, based on the currently accepted equations of Wave Mechanics and running as close as possible to Classical Dynamics.

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